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PHYSICS AND  
MATHEMATICS

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THERMONUCLEAR REACTION RATES  
IN AN ELECTRICAL DISCHARGE

Harold Grad

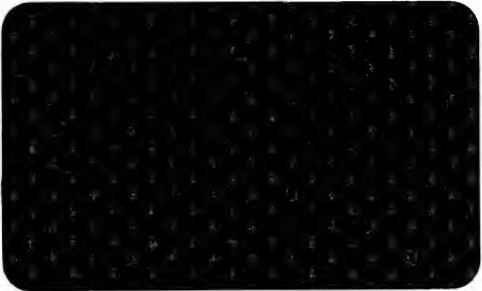
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Abstract

The ion distribution function in a plasma carrying current (e.g. in a pinch discharge) is not Maxwellian, and this fact must be taken into account in computing the thermonuclear reaction rate. In particular, it is the (non-)Maxwellian tail of the distribution which is vital to this computation. However, the conventional method of computing the distribution is legitimate only for the bulk of the distribution. This is good enough for the computation of transport coefficients but not for the computation of thermonuclear output. An asymptotic expression is derived for the tail of the ion distribution; the approximation is valid when the ratio of electron to ion mass is considered to be small and the electric field is not too large. The tail end of the distribution is found to depend exponentially on the value of the electric field, and depending on the numerical values, this result can deviate slightly or enormously from the Maxwell-averaged reaction rate. However, since the reaction rate also varies very rapidly with temperature, an estimation of plasma temperature from observed values of neutron emission will not be greatly in error even if the computation is based on the incorrect Maxwellian distribution. It should also be kept in mind that an experimental temperature deduced in this way will be the maximum that is attained even if it exists only in a very small region of the plasma.



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## THERMONUCLEAR REACTION RATES IN A DISCHARGE

Harold Grad

1. Introduction .

The problem of evaluating the electron distribution (and from it the conductivity) of a plasma which carries a current induced by an applied electric field has been treated by Landshoff [2] using the Chapman-Enskog approach to the Boltzmann equation and by Cohen, Spitzer, Routly, and Harm [3] using a Fokker-Planck equation. Exactly the same methods could be used to obtain the ion distribution needed to compute thermonuclear reaction rates. However, it is easy to verify that these methods yield a good approximation to the distribution function only for the middle range, i.e. for ions which do not greatly exceed the mean ion speed. A symptom of this defect is that the distribution function obtained by these methods is not necessarily positive. One can expect these results to be valid up to a maximum velocity which will vary as  $1/E$ . If  $E$  is small enough, the validity of this approximation will extend out to the range where thermonuclear reactions occur; however, if, as is frequently the case, the electrical discharge is used to heat and compress the plasma, the electric field will be as large as is experimentally feasible, and a better theory is probably necessary.

The theory presented here is also a linear one, applicable to

small values of the electric field, but it is a different linearization than the one adopted by Landshoff and Spitzer et al. Its purpose is to obtain a uniform approximation to the distribution function valid for large velocity as well as small. The extension of these results to larger values of  $E$  (i.e. the solution of the nonlinear problem) is being studied.

A simple example will illustrate some of these possibilities (but not necessarily in the way in which they occur in the problem at hand). Consider the ordinary differential equation  $du/dx = Eu$ . The solution is  $u = u_0 e^{Ex}$ . The linearization  $u = u_0 + Ev$  leads in a conventional way to the equation  $dv/dx = Eu_0$  with solution  $v = Eu_0 x$ . This is evidently the first term of the power series expansion of the true solution and is valid only for small values of  $Ex$ . The significant point is that, even in a linear equation in which a parameter  $E$  occurs linearly, the solution is not necessarily linear in  $E$ ; the solution is linear if  $E$  multiplies an inhomogeneous term. In the conventional linearizations of the Boltzmann and Fokker-Planck equations which were just mentioned, not only is the quadratic collision term linearized but the already linear term  $E \cdot \frac{\partial f}{\partial \xi}$  is altered to  $E \cdot \frac{\partial f^{(0)}}{\partial \xi}$  where  $f^{(0)}$  is the zero-order (Maxwellian) approximation to the distribution function  $f$ . We propose to keep the term  $E \cdot \frac{\partial f}{\partial \xi}$  unaltered. Also, in the collision term we find it possible to introduce a linearization which assumes only that there is a small departure from the Maxwellian distribution but not that the departure is a small fraction of  $f^{(0)}$ . This is essential in approximating to the tail end of the distribution where  $f$  can be strongly non-Maxwellian (but still small) even though  $E$  is small. That this can be done is a consequence of the precise form taken by the collision term.

In this context, an illuminating example is the nonlinear equation  $du/dx = Eu - u^2$ . The exact solution is  $u = Eae^{Ex}/(1+ae^{Ex})$ . For  $E = 0$  we have  $u = 1/(x+b)$ , and the usual linearization is clearly worthless for large  $x$ . However, if we happen to know or guess that  $u$  approximates  $E$  closely as  $x$  approaches infinity, the linearization  $u = E + v$  where  $v \ll E$  gives the correct asymptotic behavior at infinity.

In this report we have chosen the Fokker-Planck description rather than the Boltzmann description of a plasma for mathematical simplicity. Even though they are in some heuristic sense "almost the same" equation (see section 2), for our purposes the Fokker-Planck equation seems to have distinct mathematical advantages. Physically, the motion of a particle (electron or ion) in a plasma is governed by the superposition of many small simultaneous random forces rather than, as in ordinary kinetic theory, by a time sequence of distinct finite impulses. A particle with a given velocity sees a directed frictional force tending to bring it to rest and at the same time a random dispersional force directed "outward". The Maxwell distribution results from a balance between these two effects. In a state of non-equilibrium not only will the distribution function be altered but the friction and dispersion coefficients will also be perturbed.

A result which is, at least at first glance, surprising is the slight extent to which the ion distribution is altered from a Maxwellian by the presence of an electric field. Qualitatively, this may be seen as follows. Each particle sees a force of magnitude  $teE$  and, in addition, a frictional force. If a steady state is postulated, the mean frictional force seen by the whole collection of electrons (ions)

must be  $+eE$  ( $-eE$ ). This mean frictional force is, of course, ascribable only to the alteration of the friction coefficient concomitant with the non-Maxwellian perturbation of the distribution function since the total frictional force vanishes in equilibrium. In this momentum balance we need consider the friction on an electron due to the ion gas only, and vice versa. For the electrons, the frictional force perturbation is a definite function of velocity, and the difference between this function and its mean value (viz.  $eE$ ) yields the deviation of the distribution function from Maxwellian. On the other hand, because of the great disparity between the two masses, the ion velocity distribution is almost a  $\delta$ -function when measured on the scale of the mean electron speed. Consequently, the friction acting on an ion due to the perturbation of the electron distribution function is, to a certain degree of approximation the same for all ions. Since the mean value of this friction must be  $eE$ , and the friction is a constant we conclude that for each ion the applied force,  $-eE$ , is exactly cancelled by the additional friction caused by the perturbation of the electron distribution. Of course, this is true only to a certain order in the mass ratio.

It is interesting to note that this problem can be formulated without allowing variation in time only in the linear case, for small  $E$ . In general there is the Ohmic heating term,  $E \cdot J \sim E^2$  which precludes any simple stationary solution. There is a special case, viz. with a time-dependent electric field  $E \sim t^{-2/3}$ , which yields a similarity solution in which the temperature  $T \sim t^{2/3}$  and the distribution function scales as  $T^{-3/2} f(\xi/\sqrt{T})$ . This solution has not yet been developed.

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## 2. The Fokker-Planck Equation

The applicability of the Boltzmann equation to a plasma is a considerably more subtle affair than to an ordinary gas. The reason is the presence of long range Coulomb forces. Without taking into account any cancellation of forces, one concludes that the force due to distant charged particles overwhelms that due to neighboring particles. This is in fact the case if all particles have the same sign but must be modified in a plasma which is approximately neutral. The first step is to recognize the effect of the distant particles as being essentially macroscopic and thus to be included in Maxwell's equations as a charge density and current. The separation between "nearby" and "distant" is not arbitrary and must be taken as the range over which a significant two-particle correlation exists. This correlation length scale is the Debye length<sup>1</sup>

$$(2.1) \quad d^2 = \frac{\chi kT}{\sum n_r e_r^2} \quad ;$$

$n_r$  is the number density of the  $r$ 'th species and  $e_r$  its charge; if the ions are all singly charged,

$$(2.2) \quad d^2 = \frac{\chi kT}{ne^2}, \quad n = \sum n_r.$$

For future reference, we introduce the particle diameter (i.e. mean distance of closest approach)

$$(2.3) \quad \delta = \frac{e^2}{12\pi\chi kT},$$

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<sup>1</sup>We use mks rationalized units except where otherwise stated;

$\chi = (36\pi \times 10^9)^{-1}$  is the permittivity of free space.

the mean intermolecular distance,

$$(2.4) \quad \lambda = \frac{1}{n^{1/3}}$$

and the mean-free-path<sup>1</sup>

$$(2.5) \quad \Lambda = \Lambda^* / \log(d/\delta), \quad \Lambda^* = \frac{1}{\sqrt{2} \pi n \delta^2}.$$

Under most circumstances,

$$(2.6) \quad \delta \ll \lambda \ll d \ll \Lambda,$$

and a particularly useful relation to remember is that the Debye length is essentially the geometric mean of the particle diameter and the mean-free-path.

Although the correlation argument satisfactorily takes care of the distant (continuum) effects, it does not reduce the nearby (collisional) problem to conventional kinetic theory. The reason is that under the most interesting circumstances the Debye length is large compared to the intermolecular separation distance and this fact would seem to eliminate conventional binary collision analysis.

To resolve this difficulty a new transition length somewhere between  $\delta$  and  $\lambda$  is introduced, say  $\ell$ . For distances smaller than  $\ell$ , binary collision analysis is valid and the Boltzmann collision term is obtained. In the remaining range,  $\ell < x < d$ , a particle sees a superposition of many small forces. They are small in the sense that each one alone would produce a small deflection in the particle's path ( $x \gg \delta$ ). Following Chandrasekhar [1] and Spitzer [3], we interpret

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<sup>1</sup>Some of these numerical factors are somewhat arbitrary.

this situation stochastically as a Markoff process in which each of the particles in the range  $\ell < x < d$  contributes an independent deflection to the given test particle; to justify the Markoff assumption there is also assumed to be no correlation over times longer than the mean time required to traverse the distance  $d$ . It was pointed out by Spitzer that this stochastic analysis leads to the same result as the otherwise unjustified extension of binary collision analysis out to the distance  $d$ . The reason is that the Boltzmann equation itself represents a Markoff process. On the one hand (Boltzmann) we compute the probability distribution of binary deflections and interpret them as independent successive impulses. On the other hand (Fokker-Planck) we compute the same probability distribution of binary deflections and interpret them as simultaneous independent impulses. Since the mathematical result depends only on the probability distribution and not on the interpretation, the Boltzmann and Fokker-Planck results must agree (in the range  $\ell < x < d$ ).

From this discussion we see that the Boltzmann equation is somewhat more accurate than the Fokker-Planck equation in that it includes large-angle as well as grazing collisions; the quantity  $\log(d/\delta)$  compared to 1 roughly measures the importance of grazing collisions as compared to large angle collisions. However, in choosing a formulation for a given problem, one must also consider the mathematical feasibility of using one rather than the other.

The Fokker-Planck equation is derived from the point of view of the basic Markoff process in [3] by Spitzer et al. and in more completeness in [5] by Rosenbluth et al. It is illuminating to indicate a derivation directly from the Boltzmann equation. For simplicity,



consider a simple gas,

$$(2.7) \quad \frac{\partial f}{\partial t} + \xi_r \frac{\partial f}{\partial x_r} + \frac{\partial}{\partial \xi_r} (\chi_r f) = K \int \frac{1}{v^3} (f' f'_1 - f f_1) \frac{\sin \theta}{\cos^3 \theta} d\theta d\epsilon d\xi_1$$

$$(2.8) \quad K = \frac{e^4}{4\pi^2 \chi^2 m^2}.$$

The notation is the conventional one;  $V$  represents the relative velocity,  $\xi_1 = \xi$ , and  $f' = f(\xi')$ ,  $f'_1 = f(\xi'_1)$  refer to the velocities which result from a collision between particles with initial velocities  $\xi$ ,  $\xi_1$ . Specifically,

$$(2.9) \quad \begin{cases} \xi' = \xi + \alpha(\alpha \cdot V), \\ \xi'_1 = \xi_1 - \alpha(\alpha \cdot V) \end{cases} \quad \alpha \cdot V = V \cos \theta$$

where  $\alpha$  is a unit vector defined by the colatitude  $\theta$  and longitude  $\epsilon$  taking  $V$  as the polar axis.

For grazing deflections, the differences  $\xi' - \xi$  and  $\xi'_1 - \xi_1$  are small and we may expand the difference  $f' f'_1 - f f_1$  in a Taylor series using

$$(2.10) \quad \begin{cases} f' = f + V \cos \theta (\alpha \cdot \nabla) f + \frac{1}{2} V^2 \cos^2 \theta (\alpha \cdot \nabla)^2 f + \dots \\ f'_1 = f_1 - V \cos \theta (\alpha \cdot \nabla_1) f_1 + \frac{1}{2} V^2 \cos^2 \theta (\alpha \cdot \nabla_1)^2 f_1 + \dots \end{cases}$$

Carrying out the  $\epsilon$  integration, we get

$$(2.11) \quad \int (f' f'_1 - f f_1) d\epsilon = \pi \cos^2 \theta \left\{ \left( 1 + \frac{3}{2} \sin^2 \theta \right) (V \cdot \nabla - V \cdot \nabla_1) f f_1 \right. \\ \left. + \left( 1 - \frac{3}{2} \sin^2 \theta \right) (V \cdot \nabla - V \cdot \nabla_1)^2 f f_1 + \dots \right\}.$$

Here  $\nabla = \partial/\partial \xi$  and  $\nabla_1 = \partial/\partial \xi_1$ . The  $\theta$  integration diverges. This difficulty can be avoided either by using the correct Debye shielding

potential instead of the Coulomb potential<sup>1</sup> or by the conventional expedient of introducing a cutoff at an angle  $\theta_0$  which corresponds to the deflection produced by a particle approaching with mean relative velocity at a separation  $d$ . The result is

$$(2.12) \quad \cos \theta_0 = \delta/d.$$

Two integrals arise,

$$\int_0^{\theta_0} \cos^2 \theta \frac{\sin \theta}{\cos^3 \theta} d\theta = -\log |\cos \theta_0|$$

$$\int_0^{\theta_0} \cos^2 \theta \sin^2 \theta \frac{\sin \theta}{\cos^3 \theta} d\theta = -\log |\cos \theta_0| - \frac{1}{2}.$$

The  $1/2$  is dropped compared to the dominant term  $\log(d/\delta)$  since equivalent terms have already been dropped by the elimination of large angle deflections. Moreover, the third and higher derivatives in the Taylor series employed above can be ignored only by using this rule. After many integrations by parts, we obtain the result

$$(2.13) \quad \frac{\partial f}{\partial t} + \xi_r \frac{\partial f}{\partial x_r} + \frac{\partial}{\partial \xi_r} (X_r f) = \frac{1}{2} \frac{\partial^2}{\partial \xi_r \partial \xi_s} (b_{rs} f) - \frac{\partial}{\partial \xi_r} (a_r f)$$

where

$$(2.14) \quad b_{rs} = \pi K \log\left(\frac{d}{\delta}\right) \int \frac{v^2 \delta_{rs} - v_r v_s}{v^3} f_1 d\xi_1$$

$$a_r = 2\pi K \log\left(\frac{d}{\delta}\right) \int \frac{v_r}{v^3} f_1 d\xi_1$$

We recognize  $a_r$  to be the friction coefficient and  $b_{rs}$  the dispersion coefficient. A particle which is known to have the velocity  $\xi$  at time

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<sup>1</sup>R. Liboff, report to appear.

zero "diffuses" in the interval  $dt$  into a normal distribution with mean value  $\xi_r + a_r dt$  and second moments  $b_{rs} dt$ .

For a mixture of electrons and several sorts of ions, we quote the results

$$(2.15) \quad \frac{\partial f^i}{\partial t} + \xi_r \frac{\partial f^i}{\partial x_r} + \frac{\partial}{\partial \xi_r} (X_r^i f^i) = \frac{1}{2} \frac{\partial^2}{\partial \xi_r \partial \xi_s} (b_{rs}^i f^i) - \frac{\partial}{\partial \xi_r} (a_r^i f^i) \equiv J^i$$

where

$$(2.16) \quad a_r^i(\xi) = L^i \sum_j \frac{m^i + m^j}{m^j} A_r^j(\xi)$$

$$(2.17) \quad b_{rs}^i(\xi) = L^i \sum_j B_{rs}^j(\xi)$$

$$(2.18) \quad L^i = \frac{e^4}{4\pi\chi m_i^2} \log\left(\frac{d}{\delta}\right)$$

$$(2.19) \quad A_r^j(\xi) = \int \frac{v_r}{v^3} f^j(\eta) d\eta \quad (v_r = \eta_r - \xi_r)$$

$$= \frac{\partial}{\partial \xi_r} \int \frac{1}{v} f^j(\eta) d\eta$$

$$(2.20) \quad B_{rs}^j(\xi) = \int \frac{v_{rs}^2 \delta_{rs} - v_r v_s}{v^3} f^j(\eta) d\eta$$

$$= \frac{\partial^2}{\partial \xi_r \partial \xi_s} \int v f^j(\eta) d\eta$$

These results are identical to those given by Rosenbluth et al. in [5].

### 3. Properties of the Friction and Dispersion Coefficients

For reference we list a number of useful identities satisfied by the various coefficients:

$$(3.1) \quad \int A_r^j(\xi) f^j(\xi) d\xi = 0$$

$$(3.2) \quad \sum_i m^i \int a_r^i(\xi) f^i(\xi) d\xi = 0$$

$$(3.3) \quad \int [4\xi_r A_r^j(\xi) + B_{rr}^j(\xi)] f^j(\xi) d\xi = 0$$

$$(3.4) \quad \sum_i m_i \int [2\xi_r a_r^i(\xi) + b_{rr}^i(\xi)] f^i(\xi) d\xi = 0$$

$$(3.5) \quad \frac{\partial B_{rs}^j}{\partial \xi_s} = 2 A_r^j.$$

The first two identities are useful in deriving the momentum conservation equations and the next two are similarly related to energy conservation. The final identity will be found useful in manipulations to follow. The proofs of these identities are quite direct.

We shall find it useful to expand the indicated derivatives in the collision term,  $J_i$ , in (2.15),

$$(3.6) \quad J_r^i = \frac{1}{2} b_{rs}^i \frac{\partial^2 f^i}{\partial \xi_r \partial \xi_s} + p_r^i \frac{\partial f^i}{\partial \xi_r} - \frac{\partial Q_r^i}{\partial \xi_r} f^i$$

where

$$(3.7) \quad p_r^i = \frac{\partial b_{rs}^i}{\partial \xi_s} - a_r^i = L^i \sum_j \frac{m^j - m^i}{m^j} A_r^j$$

$$(3.8) \quad Q_r^i = a_r^i - \frac{1}{2} \frac{\partial b_{rs}^i}{\partial \xi_s} = m^i L^i \sum_j \frac{1}{m^j} A_r^j.$$

The coefficients  $P^1$  and  $Q^1$  are now expressed in terms of  $A$  alone. A significant point is that the  $P^1$  summation contains only terms with  $j \neq 1$ .

The various coefficients can be explicitly evaluated for a Maxwellian  $f$  (cf. [1]), viz.

$$(3.9) \quad f^{(0)} = \frac{n}{(2\pi RT)^{3/2}} e^{-\xi^2/2RT}, \quad R = k/m$$

(the superscript 1 has been omitted). By introducing  $V$  as the variable of integration instead of  $\eta$  and introducing polar coordinates with the vector  $\xi$  as the polar axis we obtain

$$(3.10) \quad \begin{cases} \int \frac{1}{V} f^{(0)}(\eta) d\eta = \frac{n}{\sqrt{RT}} F\left(\frac{\xi}{\sqrt{RT}}\right) \\ \int V f^{(0)}(\eta) d\eta = n \sqrt{RT} G\left(\frac{\xi}{\sqrt{RT}}\right) \end{cases}$$

where

$$(3.11) \quad \begin{cases} F(x) = \sqrt{\frac{2}{\pi}} \frac{1}{x} \int_0^x e^{-(1/2)y^2} dy \\ G(x) = \sqrt{\frac{2}{\pi}} \left\{ \left(x + \frac{1}{x}\right) \int_0^x e^{-(1/2)y^2} dy + e^{-(1/2)x^2} \right\} \end{cases}$$

By differentiation we then find

$$(3.12) \quad A_r^{(0)} = \frac{n}{RT} \frac{x_r}{x} F'(x)$$

$$(3.13) \quad B_{rs}^{(0)} = \frac{n}{\sqrt{RT}} \left\{ \frac{x_r^2 \delta_{rs} - x_r x_s}{x^3} G'(x) + \frac{x_r x_s}{x^2} G''(x) \right\}$$

where we have used the abbreviation

$$(3.14) \quad x_r = \frac{\xi_r}{\sqrt{RT}}.$$

For small  $\xi$  we have,

$$(3.15) \quad A_r^{(0)} = -\frac{1}{3} \sqrt{\frac{2}{\pi}} \frac{n}{RT} x_r (1 - \frac{3}{10} x^2 + \frac{3}{8} x^4 + \dots)$$

$$(3.16) \quad B_{rs}^{(0)} = \sqrt{\frac{2}{\pi}} \frac{n}{\sqrt{RT}} \left[ \delta_{rs} \left( \frac{2}{3} - \frac{1}{15} x^2 + \frac{1}{140} x^4 + \dots \right) + x_r x_s \left( -\frac{2}{15} + \frac{1}{35} x^2 + \dots \right) \right]$$

and for large  $\xi$ ,

$$(3.17) \quad A_r^{(0)} = -\frac{n}{RT} \frac{x_r}{x^3}$$

$$(3.18) \quad B_{rs}^{(0)} = \frac{n}{\sqrt{RT}} \left\{ \frac{x^2 \delta_{rs} - x_r x_s}{x^3} - \frac{x^2 \delta_{rs} - 3x_r x_s}{x^5} \right\};$$

the terms neglected are in this case exponentially small.

In the presence of an electric field  $f$  is no longer Maxwellian. For a perturbation which is linear in  $E$  we have

$$(3.19) \quad f(\xi) = f^{(0)}(\xi) + E \cdot \xi \phi\left(\frac{1}{2} \xi^2\right).$$

By the assumption of linearity,  $\phi$  is a scalar function of the vector  $\xi$  alone (not of  $E$ ) and is therefore a function of only the magnitude of  $\xi$ . For the perturbations, defined by

$$(3.20) \quad \begin{cases} A_r = A_r^{(0)} + A_r^{(1)} \\ B_{rs} = B_{rs}^{(0)} + B_{rs}^{(1)} \end{cases}$$

we have

$$(3.21) \quad A_{\mathbf{r}}^{(1)} = E_s \int \frac{v_r \eta_s}{v^3} \phi\left(\frac{1}{2} \eta^2\right) d\eta$$

and

$$(3.22) \quad B_{\mathbf{rs}}^{(1)} = E_t \int \frac{v^2 \delta_{\mathbf{rs}} - v_r v_s}{v^3} \eta_t \phi\left(\frac{1}{2} \eta^2\right) d\eta$$

We are particularly interested in the values of  $A^{(1)}$  and  $B^{(1)}$  for small  $\xi$ . To this end we expand

$$(3.23) \quad \begin{aligned} \phi\left(\frac{1}{2} \eta^2\right) &= \phi\left(\frac{1}{2} v^2 + v \cdot \xi + \frac{1}{2} \xi^2\right) \\ &= \phi\left(\frac{1}{2} v^2\right) + v \cdot \xi \phi'\left(\frac{1}{2} v^2\right) + \frac{1}{2} \xi^2 \phi'\left(\frac{1}{2} v^2\right) + \frac{1}{2} (v \cdot \xi)^2 \phi''\left(\frac{1}{2} v^2\right) \\ &\quad + \frac{1}{2} \xi^2 v \cdot \xi \phi''\left(\frac{1}{2} v^2\right) + \frac{1}{6} (v \cdot \xi)^3 \phi'''\left(\frac{1}{2} v^2\right) + \dots \end{aligned}$$

obtaining, after some manipulation

$$(3.24) \quad A_{\mathbf{r}}^{(1)} = \frac{4}{3} \pi \phi_1 E_{\mathbf{r}} - \frac{2}{5} \pi \phi_0 (E_{\mathbf{r}} \xi^2 + 2 E \cdot \xi \xi_{\mathbf{r}}) + \dots$$

$$(3.25) \quad \begin{aligned} B_{\mathbf{rs}}^{(1)} &= \frac{8}{15} \pi \phi_1 (E \cdot \xi \delta_{\mathbf{rs}} + E_{\mathbf{r}} \xi_s + E_s \xi_{\mathbf{r}}) \\ &\quad - \frac{4}{35} \pi \phi_0 [2 E \cdot \xi \xi_{\mathbf{r}} \xi_s + \xi^2 (E_{\mathbf{r}} \xi_s + E_s \xi_{\mathbf{r}} + E \cdot \xi \delta_{\mathbf{rs}})] \\ &\quad + \dots \end{aligned}$$

where we have set

$$(3.26) \quad \begin{cases} \phi_0 &= \phi(0) \\ \phi_1 &= \int_0^\infty \phi(x) dx \end{cases} .$$

#### 4. Reduction to Spherical Coordinates

In our problem the external force term,  $X_r$  in equation (2.15), is the electric field,

$$(4.1) \quad X_r^1 = \frac{e^1}{m^1} E_r, \quad e^1 = \pm e.$$

We can use spherical or cylindrical coordinates; we choose spherical coordinates  $(r, \theta)$  with the direction of  $E$  as the polar axis,

$$(4.2) \quad \begin{cases} \xi^2 = r^2 \\ E \cdot \xi = Er \cos \theta. \end{cases}$$

In order to transform the collision term in the form (3.6) we note that

$$(4.3) \quad \Delta = \frac{\partial^2}{\partial \xi_r \partial \xi_r} = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta};$$

also

$$(4.4) \quad \begin{cases} \xi_r \frac{\partial}{\partial \xi_r} = r \frac{\partial}{\partial r} \\ \xi_r \xi_s \frac{\partial^2}{\partial \xi_r \partial \xi_s} = r^2 \frac{\partial^2}{\partial r^2} \\ E_r \frac{\partial}{\partial \xi_r} = E (\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}) \\ (\xi_r \xi_s + E_s \xi_r - 2E \cdot \xi \frac{\xi_r \xi_s}{\xi^2}) \frac{\partial^2}{\partial \xi_r \partial \xi_s} = -2E \sin \theta (\frac{\partial^2}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial}{\partial \theta}) \end{cases}$$

Next we set

$$(4.5) \quad \begin{cases} b_{rs} = b_1 \frac{\xi_r \xi_s}{\xi^2} + b_2 (\delta_{rs} - \frac{\xi_r \xi_s}{\xi^2}) + b_3 (E_r \xi_s + E_s \xi_r - 2E \cdot \xi \frac{\xi_r \xi_s}{\xi^2}) \\ P_r = p_1 \xi_r + p_2 E_r \\ Q_r = q_1(r) \xi_r + q_2(r) E \cdot \xi \xi_r + q_3(r) E_r \end{cases}$$



and compute

$$(4.6) \quad \frac{\partial Q_r}{\partial \xi_r} = 3q_1 + r \frac{\partial q_1}{\partial r} + E \cos \theta (4rq_2 + r^2 \frac{\partial q_2}{\partial r} + \frac{\partial q_3}{\partial r}).$$

Note that  $b_{rs}$  is separated into three orthogonal components in the sense that the product of  $\xi_r \xi_s / \xi^2$  and  $\delta_{rs} - \xi_r \xi_s / \xi^2$  is zero, etc.

Dropping the time and space derivatives in the Fokker-Planck equation, we obtain

$$(4.7) \quad \begin{aligned} & \frac{1}{2} b_1 \frac{\partial^2 f}{\partial r^2} - b_3 E \sin \theta \frac{\partial^2 f}{\partial r \partial \theta} + \frac{1}{2} b_2 \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \\ & + \left[ \frac{1}{r} b_2 + r p_1 + E \cos \theta (p_2 - \frac{e}{m}) \right] \frac{\partial f}{\partial r} + \\ & + \left[ \frac{1}{2} b_2 \cot \theta + E \sin \theta (b_3 - p_2 + \frac{e}{m}) \right] \frac{1}{r} \frac{\partial f}{\partial \theta} - \\ & - \left[ 3q_1 + r \frac{\partial q_1}{\partial r} + E \cos \theta (4rq_2 + r^2 \frac{\partial q_2}{\partial r} + \frac{\partial q_3}{\partial r}) \right] f = 0; \end{aligned}$$

the superscript  $i$  has been omitted.

## 5. Solution for Thermal Speeds

In order to carry out the asymptotic evaluation of  $f$  for large  $\xi$  in the next section, it is necessary to first discuss the behavior of the solution for moderate values of  $\xi$ . In this range we can use the conventional linearization (3.19). It will not be necessary to solve this problem, only to estimate the order of magnitude of the solution. Substituting

$$(5.1) \quad f = f^{(0)} + f^{(1)} = f^{(0)} + E \cos \theta \, r \, \phi$$

into equation (4.7) and linearizing with respect to  $E$  we obtain

$$(5.2) \quad \begin{aligned} & \frac{1}{2} b_1^{(0)} \frac{\partial^2 f^{(1)}}{\partial r^2} + \frac{1}{2} b_2^{(0)} \frac{1}{r^2} \frac{\partial^2 f^{(1)}}{\partial \theta^2} + \left( \frac{1}{r} b_2^{(0)} + r p_1^{(0)} \right) \frac{\partial f^{(1)}}{\partial r} \\ & + \frac{1}{2} b_2^{(0)} \cot \theta \frac{1}{r} \frac{\partial f^{(1)}}{\partial \theta} - \left( 3 q_1^{(0)} + r \frac{\partial q_1^{(0)}}{\partial r} \right) \\ & = - \left\{ \frac{1}{2} b_1^{(1)} \left( \frac{r^2}{R^2 T^2} - \frac{1}{RT} \right) - \left( \frac{1}{r} b_2^{(1)} + r p_1^{(1)} \right) \frac{r}{RT} + E \cos \theta \left( \frac{e}{m} - p_2 \right) \frac{r}{RT} \right. \\ & \quad \left. - \left[ 3 q_1^{(1)} + r \frac{\partial q_1^{(1)}}{\partial r} + E \cos \theta \left( 4 r q_2^{(0)} + r^2 \frac{\partial q_2^{(0)}}{\partial r} + \frac{\partial q_3^{(0)}}{\partial r} \right) \right] \right\} f^{(0)}. \end{aligned}$$

The inhomogeneous terms have been collected on the right hand side. It is easy to verify that  $E \cos \theta$  cancels out as a common factor leaving an ordinary differential equation for  $\phi \left( \frac{1}{2} r^2 \right)$ . We wish to estimate the magnitude of the various inhomogeneous terms considering that  $m^-/m^+ = m/M$  is small. The given equation (5.2) is supposed to be that for the ions. Consequently, we are interested in a range of

velocities  $r$  comparable to  $\sqrt{kT/M}$ . The dominant term on the right (this arises from the original term  $E \cdot \partial f / \partial \xi$ ) is

$$E \cos \theta \frac{e}{M} \frac{r}{RT}.$$

Now, following Spitzer, we can evaluate the parameter  $p_2$  which subtracts from this dominant term by referring to the conservation of momentum. It is easiest to use the form (2.15). Multiplying through the ion equation by  $\xi_r$  and integrating over velocity yields

$$\int \left( \frac{e}{M} E_r + a_r^+(\xi) \right) f^+(\xi) d\xi = 0.$$

Using (2.16) and the identity (3.1) for  $A^+$  we are left with

$$\frac{e}{M} E_r n^+ + L^+ \frac{M+m}{m} \int A_r^-(\xi) f^+(\xi) d\xi = 0.$$

Linearizing  $A^-$  and  $f^+$  in the integrand, we find that the  $A^{(0)} f^{(1)}$  term is small of order  $m/M$  compared to the  $A^{(1)} f^{(0)}$  term and, using the first term in the expansion (3.24) (the whole ion distribution is at the very low velocity end of the electron distribution) we obtain

$$(5.3) \quad \phi_1^- = - \frac{3}{4\pi} \frac{em}{M^2 L^+}$$

Now, evaluating  $P_r^+$  from (3.7) and (3.24) and from this the coefficient  $p_2$  (cf. (4.5)), we find

$$\begin{aligned} P_r^+ &= L^+ \frac{m-M}{m} A_r^- \\ &\sim - \frac{4}{3}\pi \phi_1^- L^+ \frac{M}{m} E_r \\ (5.4) \quad p_2^+ &= - \frac{4}{3}\pi \phi_1^- L^+ \frac{M}{m} = \frac{e}{M}. \end{aligned}$$

We see then that what we would expect to be the dominant term in the equation drops out, the reason being the one described in the introduction, namely, the cancellation of the applied  $E$  by the perturbation of the ion-electron friction coefficient. It is very easy to verify that the other inhomogeneous terms in equation (5.2) are also small, on the order of  $m/M$ ; the coefficients  $b_1^{(1)}$ , etc. are obtained by using the low velocity approximation to the electron coefficients derived in section 3. It would be possible to solve (5.2) just as in [3] but this will not be necessary. As it turns out, only the coefficient  $\phi_1^-$  of the perturbed electron distribution is needed (and this has just been evaluated in (5.3)). The perturbation of the ion distribution in the thermal range can be ignored to low order in an expansion in  $m/M$ .

## 6. Asymptotic Expansion of the Tail of the Distribution

We introduce as an expansion parameter the mass ratio,

$$(6.1) \quad \begin{cases} \frac{m}{M} = \epsilon^6 \\ \frac{C^+}{C^-} = \frac{\sqrt{R^+T}}{\sqrt{R^-T}} = \sqrt{\frac{m}{M}} = \epsilon^3 \end{cases}$$

For deuterons  $M/m \sim 3600$ ,  $C^-/C^+ \sim 60$ , and  $\epsilon \sim 1/4$ . Actually we have two small parameters,  $E$  and  $\epsilon$ , but they will eventually be related. We are interested in a range of ion velocities large compared to  $C^+$  and small compared to  $C^-$ . In this intermediate range the Maxwellian friction coefficient  $a^+$  has the approximate form (from (2.16), (3.15), (3.17))

$$(6.2) \quad \begin{aligned} a^+ &= a^{++} + a^{+-} \\ &= 2L^+A^+ + \frac{M}{m} L^+A^- \\ a^+ &\sim -\frac{nL^+}{R^+T} \left( \frac{2}{x^2} + \frac{1}{3} \right) \sqrt{\frac{2}{\pi}} \epsilon^3 x \end{aligned}$$

where

$$(6.3) \quad x = \frac{\xi}{C^+}$$

and the absolute magnitude of the vector  $a^+$  has been taken. The minimum value of  $a^+$  is found at

$$(6.4) \quad \epsilon x = (72\pi)^{1/6} \sim 2.5.$$

At this point the value of the friction coefficient is reduced by a factor on the order of  $\epsilon^2$  times the peak values attained near  $\xi \sim C^+$  and  $\xi \sim C^-$ .

It is not clear, a priori, whether we are interested in the range  $x \sim 1/\epsilon$  or  $x \sim 1/\epsilon^2$  (friction  $\sim \epsilon^2$  or  $\epsilon$ ). Presumably the correct scaling depends on which range of velocities produces the thermonuclear reactions in a given experiment; however, since there is only a factor of four at stake, mathematical expedience would seem to be the predominant criterion.

Several important introductory remarks will be made now. First, for small  $E$ , even though the tail end of the distribution may be extremely non-Maxwellian it will still die down exponentially. Now, the friction and dispersion coefficients are given as integrals over the whole distribution. They will therefore be determined (even the values at high velocities) by the middle range of the distribution for which the conventional linearization (3.19) is valid. The use of this linearization in computing the perturbations (3.24) and (3.25) is in this way justified. For those coefficients which are integrated over the electron distribution, we can use Spitzer's determination of this distribution. For the ion coefficients we can take the unperturbed Maxwellian values to the order which will be kept in the expansion (cf. section 5).

The reason why the present linearization gives correct results even if the perturbation is large compared to  $f^{(0)}$ , provided only that it is small in an absolute sense, is that instead of expanding a quadratic functional

$$Q(f, f) = Q(f^{(0)}, f^{(0)}) + Q(f^{(0)}, f^{(1)}) + Q(f^{(1)}, f^{(0)}) + Q(f^{(1)}, f^{(1)})$$

and then dropping  $Q(f^{(1)}, f^{(1)})$  compared to  $Q(f^{(0)}, f^{(1)})$ , we find it possible to expand

$$Q(f, f) = Q(f, f^{(0)}) + Q(f, f^{(1)})$$

where the  $f^{(1)}$  has been found independently and is not an unknown.

Adopting these approximations, we obtain the following equation valid in the range  $C^+ \ll \xi \ll C^-$  after factoring out the Maxwellian,

$$(6.5) \quad f = f^{(0)} g,$$

$$(6.6) \quad \begin{aligned} & \frac{1}{3} \sqrt{\frac{2}{\pi}} \epsilon^3 \frac{\partial^2 g}{\partial r^2} - \frac{8}{15} \pi \frac{\phi_1}{n} C^+ E \sin \theta \frac{\partial^2 g}{\partial r \partial \theta} + \frac{1}{2} \frac{C^+}{r^3} \frac{\partial^2 g}{\partial \theta^2} \\ & - \frac{1}{3} \sqrt{\frac{2}{\pi}} \epsilon^3 \frac{r}{(C^+)^2} \frac{\partial g}{\partial r} + \frac{1}{2} \frac{C^+}{r^3} \cot \theta \frac{\partial g}{\partial \theta} + \frac{4}{5} \pi \frac{\phi_1}{n} \frac{r^3}{(C^+)^3} E \cos \theta g = 0 \end{aligned}$$

It is easy to see by inspection that taking  $E \sim \epsilon$  and  $r \sim 1/\epsilon$  yields a solution for  $g$  which differs by only a small amount from unity; the conventional linearization would therefore apply out to this range; (we must keep in mind that  $\phi_1$  has order  $\epsilon^6$ ). On the other hand, taking  $r \sim 1/\epsilon^2$ , we see that the terms in  $\partial g / \partial r$  and  $g$  are dominant and we obtain the asymptotic solution

$$(6.7) \quad g = \exp \left\{ - \frac{12}{5} \pi \gamma_{4\pi} \frac{\chi^2_{kT}}{n \epsilon^3 \log(d/\delta)} \left( \frac{\epsilon^2 r^2}{2R^+ T} \right)^{3/2} E \cos \theta \right\}.$$

The exponent has order unity when the energy gained by a particle accelerating under  $E$  for a mean-free-path is comparable to  $kT$ .

For  $E \sim \epsilon$  and  $r \sim 1/\epsilon^2$  the exponent in (6.7) has order  $1/\epsilon^2$ . This should be compared with the exponent in the Maxwellian which would then be of order  $1/\epsilon^4$ . Clearly, for  $\epsilon$  small, the perturbation is a very large factor applied to the Maxwellian, but the Maxwellian

is extremely small to an even higher order. The asymptotic expansion for  $g$  has been carried to only one significant term; the factor  $g$  is not accurate to this order but its logarithm is. Fortunately this is sufficient to obtain interesting information.

From the form taken by  $g$  in (6.7) it is quite clear how to proceed to get a uniform approximation to  $f$  for thermal as well as large velocities. The linear perturbation  $f = f^{(0)} + E r \cos \theta \phi(\frac{1}{2}r^2)$  has been seen to be valid up to  $r \sim 1/\epsilon$ . But, so long as the perturbation is small, this is entirely equivalent to  $f = f^{(0)} \exp \{E \cos \theta \psi(r)\}$  where  $r\phi = f^{(0)}\psi$ . This exponential form is therefore valid for the whole range of  $r$  (up to  $r \sim 1/\epsilon^2$ ). If it is substituted into the Fokker-Planck equation and the result linearized with respect to  $E$ , an ordinary differential equation for  $\psi(r)$  will result, similar to the one solved by Spitzer [3]. This analysis will not be carried out here since the asymptotic formula is probably sufficiently accurate for the present purpose. It should be noted that although this asymptotic expansion can be continued, computing additional higher order terms, there will remain an undetermined constant of integration (multiplicative factor on  $g$ ). This constant can only be determined by matching the asymptotic solution to the low velocity result, i.e. by carrying out the uniformly valid procedure just mentioned.

It should be relatively easy to detect the presence of the phenomenon described by equation (6.7) because of the remarkable fact that the deviation from Maxwellian is opposite to what would be expected from the direction of  $E$ . This is not surprising when we recall that the presumably dominant term in the Fokker-Planck equation,  $E \cdot \frac{\partial f}{\partial \xi}$ , was cancelled by the perturbation of the electron friction coefficient.



## 7. Thermonuclear Output

We can write the asymptotic expression for  $g$ , equation (6.7),

as

$$(7.1) \quad g = \exp \left\{ -\alpha(E \cdot \xi) \xi^2 \right\}$$

where

$$(7.2) \quad \alpha = \frac{12}{5} \pi^{3/2} \frac{\chi^2}{n e^3 \log(d/\delta)} \frac{\epsilon^3 M^{3/2}}{\sqrt{2kT}}.$$

To dominant (i.e. exponential) order, the D-D cross-section takes the form

$$(7.3) \quad \sigma \sim \exp \left\{ \mu \left( \frac{1}{2} M V^2 \right) \right\}.$$

To dominant order, the average cross-section, i.e. the thermonuclear output, is given by

$$(7.4) \quad \begin{aligned} \nu &= \int \exp \left\{ \mu \left( \frac{1}{2} M V^2 \right) - \frac{\frac{1}{2} M \xi^2}{kT} - \alpha(E \cdot \xi) \xi^2 - \frac{\frac{1}{2} M \xi_1^2}{kT} - d(E \cdot \xi_1) \xi_1^2 \right\} d\xi d\xi_1 \\ &= \int \exp \left\{ \mu \left( \frac{1}{2} M V^2 \right) - \frac{M}{4kT} (V^2 + W^2) - \frac{1}{4} a_{EW} (3V^2 + W^2) \right\} d\xi d\xi_1 \end{aligned}$$

where

$$(7.5) \quad W = \xi_1 + \xi, \quad V = \xi_1 - \xi.$$

To the same dominant order, the value of  $\nu$  can be approximated by maximizing the exponent with respect to  $V$  and  $W$ . This occurs at values  $V_1, W_1$  given by

$$(7.6) \quad \left\{ \begin{aligned} W_1 &\sim -\frac{3}{2} \frac{kT}{M} a_{EW}^2 \\ \mu \left( \frac{1}{2} M V_1^2 \right) &= \frac{1}{2kT} - \frac{9}{4} kT \frac{a_{EW}^2 E^2}{M^2} V_0^2 \end{aligned} \right.$$

( $V_1$  is given implicitly by the second relation). Here  $V_0$  is the maximizing value for the Maxwellian distribution, i.e.

$$(7.7) \quad \mu\left(\frac{1}{2}MV_0^2\right) = \frac{1}{2kT}.$$

Inserting (7.6) into (7.4) and treating  $E$  as a small perturbation, we obtain

$$(7.8) \quad \nu = \nu_0 \exp \left\{ \frac{9}{16} \frac{kT}{M} a^2 E^2 V_0^4 \right\}$$

where  $\nu_0$  is the Maxwellian reaction rate.

For temperatures which are not too high we have (e.g. see Thompson, [4])

$$(7.9) \quad \frac{\frac{1}{2}MV_0^2}{kT} \sim \frac{12.4}{T^{1/3}};$$

on the right, the temperature  $T (=kT)$  is in kev. Rewriting (7.8) in these units, we obtain

$$(7.10) \quad \nu = \nu_0 \exp \left\{ \left( \frac{1.5 \times 10^{15} E T^{2/3}}{n \log(d/\delta)} \right)^2 \right\},$$

where  $n$  is in ions/cm<sup>3</sup> and  $E$  is in volts/cm. For example, taking  $n = 10^{14}$ ,  $T = 0.5$  ( $5 \times 10^6$  °C.) for which  $\log(d/\delta) \sim 14$ , we find that the correction becomes appreciable for  $E \sim 1$  volt/cm. Any larger value of  $E$  will produce an enormous increase in thermonuclear output. The most striking feature of this formula is the extremely sensitive dependence on  $E$ ,  $T$ , and  $n$ . We note, however, that the perturbation from  $V_0$  to  $V_1$  is small even when  $\nu$  as computed from (7.8) is much larger than  $\nu_0$ . This means that use of the Maxwellian formula to estimate  $T$  from an experimentally observed value of  $\nu$  will not produce a large error. One must be careful, however, to interpret such a

temperature as the maximum which exists locally anywhere in the plasma.



## References

- [1] Chandrasekhar, S., Revs. Modern Phys., 15, 1 (1943).
- [2] Landshoff, R., Phys. Rev., 76, 904 (1949).
- [3] Cohen, R.S., Spitzer, L., and Routly, P., Phys. Rev., 80,  
230 (1950).  
Spitzer, L. and Harm, R., Phys. Rev., 89, 977 (1953).
- [4] Thompson, W.B., Proc. Phys..Soc., London, 70, 1 (1957).
- [5] Rosenbluth, M.N., MacDonald, W.M., and Judd, D.L., Phys. Rev.,  
107, 1 (1957).

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